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Shear Waves in Stratified Rotating Liquids

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§1. Introduction.

Meteorologists have during many decades tried to construct laboratory models of the Earth's atmospheric circulations. Since World War II these attempts have met with notable success in a number of different places and much useful information bearing on weather processes and general circulation patterns has been obtained. Of course the atmosphere cannot be modeled exactly in the laboratory in the sense of making all dimensionless scale parameters have the same value for the model as for the atmosphere. Nevertheless, it has been shown by Fultz and others [2, 3, 4, 5, 7] that if attention is focused on particular aspects of the atmospheric motions certain scale parameters are very much more important than others. For example, the Rossby number, which measures the ratio of the inertial force to the Coriolis force, is by far the most important scale parameter for the modeling of the large scale general circulation patterns.

Two of the laboratory models which permit the duplication of the Rossby number make use of rotating liquids in cylindrical vessels. Both have been very successful in duplicating the hemispheric quasi-stationary pattern of cyclones and anticyclones which is a feature of mid-latitude weather maps. In the first of these, the 'heating' model, the flow takes place in the annular region between two cylindrical surfaces kept at different temperatures, the colder surface being nearer the axis. Because of differential volume changes due to the heat transfer between

the inner and outer cylinders, a thermal wind is created which produces a continuous wind shear and vorticity change in the vertical direction. In the second or 'polar front' model, two immiscible liquids of very slightly different densities are rotated in the cylindrical pan. The velocity and vorticity are constant along verticals in each liquid separately but have a jump discontinuity at the interface between the two liquids. All of the wind shear is concentrated in the interface. The energy needed to drive the fluid in this model comes from changes in the pan rotation speed.

Both of the models described above are baroclinic, i.e., the surfaces of constant pressure do not coincide with those of constant volume, or to put it more meaningfully, the vorticity is not constant along verticals. The models are therefore capable of displaying some of the most complex features of atmospheric circulations, such as the deepening and filling of cyclonic systems, as well as the simplest features like relative wave speed and amplitude. The 'heating' model is somewhat closer to the Earth's atmosphere in its mode of operation and is also somewhat easier to manipulate experimentally. On the other hand, the 'polar front' model seemed to me to offer certain advantages from the point of view of a thorough mathematical treatment. I suspect very strongly, however, that the mathematical techniques developed here will also be of use in studying the 'heating' model.

The sequence of events which occurs in the 'polar front' model has been recorded by Fultz [5, 6, 9] but I should like to

recapitulate it here briefly because of its significance for the subsequent mathematical development. The apparatus is simplicity itself, consisting of a cylindrical glass pan mounted axially on a table whose rotation rate can be controlled over wide ranges. The liquid used for the bottom layer is a monochlorobenzene-toluene solution whose density can be adjusted to a value just slightly greater than that of the water-detergent mixture which is used for the upper layer. Because of the very small density difference (about 0.5 percent in most of the experiments) the lower liquid has very nearly zero weight. The shape of the interface between the two liquids is therefore very sensitive to small accelerations. From rest the liquids are gradually brought to a steady rotation. At the slow speeds which are used the free surface is scarcely curved but the interface is a rather steep paraboloid dished upward. The pressure gradient force at the interface is in equilibrium with the centripetal acceleration and the quite small weight of the lower liquid. The slopes are therefore of the order of 1 rather than 1/100, as they are in the atmosphere along frontal surfaces. Unless the lower layer is quite deep and the angular speed small, the denser liquid will ordinarily be forced away from the axis into an annular ring at the lower edge of the cylinder with the interface touching the bottom of the pan in a circular trace.

If radial equilibrium with essentially solid body rotations in both layers is achieved at a relatively low speed and the pan is subsequently speeded up by 20 or 30 percent, the

motions in the two layers take on quite a different aspect. Because the lower liquid adjusts itself more readily to the new rotation, whereas the upper liquid seems not to sense the change in speed, a velocity difference is created between the two layers. The first effect of this is to force the ring of dense liquid even farther away from the axis; the second effect of the shear is to produce waves, by mechanisms which are still not too well understood, upon the interface. Although the wave motions extend throughout both liquids, they are to an extent compensating, since the free surface changes relatively little from its initial paraboloidal shape of slight curvature. The number of frontal waves and the speed with which they move relative to the mass motion of the two layers depends critically upon the shear that has been introduced. The waves themselves are remarkably steady, but as the secondary frictional circulations in the two layers gradually erase the velocity difference, there are rather abrupt transitions into new modes of oscillation with an evolution toward fewer, slower moving waves of longer wave length. The late stages, as might be expected, are inclined to be complicated and to a large extent unpredictable.

The type of flow described above is called abnormal because it is essentially the opposite of motions encountered on the Earth. The so-called normal flow will be obtained in the model if the initial radial equilibrium is achieved at a relatively high rotation rate and the pan is subsequently slowed down by 20 or 30 percent. The first effect of the slowing down of the lower layer is to cause it to collect in the center

forming a dome of dense liquid on the axis. The shape of the dome in the early stages is a slightly flattened paraboloid dished downwards. The second effect of the velocity shear is again to produce waves upon the frontal surface, the number and wave speed again depending upon the amount of shear which has been introduced. A feature of the motions is the formation of a rather pronounced westerly jet or zone of high speed flow over the region of greatest slope on the interface, which occurs usually along the peripheral edge of the dome. This same phenomenon occurs, though not so noticeably, in the abnormal flow regime. Such behavior is not unexpected in view of the other evidences of a quasi-geostrophic adjustment of the velocity and pressure fields. The evolution of the motion toward fewer and longer waves is the same as before but there is an even greater tendency in the normal flow regime for nonlinear effects to come into play. These produce a distortion of the wave troughs with a steepening of the western slope and flattening of the eastern slopes, producing the familiar classic cold front - warm front picture. Eventually occlusion of the waves occurs and the motion degenerates into turbulence.

The sequence of events as outlined above is very suggestive for the mathematical development. The fact that equilibrium flows exist and are stable suggests that we should look for exact solutions of the equations of motion and boundary conditions which depend only on the radial coordinate and the height, This is carried out in Article 3. Although the equations are to an extent under-determined all the solutions have

the property that the interface may be dished either up or down depending upon the relative magnitude of the transverse velocities in the two layers and their density ratio. The most useful equilibrium solution seems to be the one discovered by Helmholtz in 1888 [1] corresponding to solid body rotations in the two layers.

Because so many features of the actual motions are explained at least partially by the solid body equilibrium solutions it seems reasonable to look for perturbation solutions which are in some sense close to the solid body rotations. The simplest method of insuring this property is to take the first variation of the full differential equations and boundary conditions and insert in them for the unvaried quantities the solid rotation solutions. The perturbation equations which I obtain in this way bear a rather striking resemblance to the equations of motion which are used in dynamical meteorology. Although the perturbation equations are linear, they are still partial differential equations in three independent variables r, θ, t and are six in number. A frontal assault on the equations to determine solutions satisfying given initial conditions seems therefore to be out of the question.

A prominent feature of the observed wave motions, however, was their remarkable stability, indicating the possibility that progressing wave type solutions might exist. I, therefore, look next for a factorization of the perturbation quantities into a form factor dependent only on the radius multiplied by a periodic function of a linear combination of angle variable and the time.

The partial differential equations are thereby reduced to coupled ordinary differential equations. One further feature of the observed waves has still to be incorporated into the mathematical theory: the compensation mechanism which permits waves of large amplitude on the interface but leaves the free surface of the upper liquid relatively unperturbed, although the motions in the upper liquid are far from negligible. On the other hand simply setting the perturbation of the free surface equal to zero is equivalent to making the perturbation of the interface also equal to zero unless the wave speeds are restricted in a certain way by setting one parameter in the theory equal to zero. The real justification for this whole procedure is to be found in the fact that when the density ratio of the two liquids is nearly unity the type of perturbation which actually occurs corresponds to a singular perturbation of the differential equations in the small parameter $\varepsilon = 1-\beta$, where β is the ratio of the density in the upper liquid to that in the lower.

After these preliminaries what is left is mathematics. The result of the assumptions above and some eliminations is an ordinary differential equation of second order for the radial dependence of the height of the interface. After some not quite obvious changes of variable the equation can be brought to the standard form of the hypergeometric differential equation. It turns out that a solution satisfying the boundary conditions at the outer wall of the cylinder and regular on the axis can exist for a given number of waves around the periphery of the cylinder only if the angular speeds in the two layers are in a certain

ratio to each other. The actual numerical value of the ratio is found from a rather complicated transcendental equation. The results are summarized in Tables I-III. The explanation of their calculation is given in Part II of this report along with graphs of the solutions and a detailed analysis of their correspondence with the observed results.

§2. Mathematical formulation of the problem.

In order that the results can be immediately applied to an analysis of the Fultz experiments, I shall derive the equation of motion in terms of cylindrical polar coordinates r, θ, z with corresponding velocity components u, v, w. The velocities will in all cases be referred to the nonrotating laboratory reference frame. In a subsequent report it will be shown how a very similar development can be carried out for the exterior of a hemisphere with gravity directed radially inward and rotations occurring about the polar axis. The two situations are in fact not greatly dissimilar since we can imagine a continuous mapping or deformation of the physical space which would leave the axis of rotation invariant but would carry the plane horizontal surfaces of the cylindrical flow over into the concentric spherical horizontal surfaces of the hemispherical flow. This basic physical similarity accounts in large part for the relevance of the Fultz experiments to the study of atmospheric circulations. Fluid dynamical likenesses are also significant, of course.

The pressure will be denoted by p and the constant density by ρ (its reciprocal, the specific volume α, will also be used occasionally). In order to distinguish the two layers of liquid I shall use a prime on all quantities referring to the upper, less dense liquid while the absence of a prime refers to the lower, denser liquid.

The boundaries of the domain are determined by the bottom surface $z = 0$, the cylinder surface $r = R$, the free surface of the upper liquid $z = \underline{\Phi}(r,\theta,t)$, and the interface $z = \phi(r,\theta,t)$ separating the two liquids. On these bounding surfaces we have the following dynamic and kinematic conditions which must be satisfied at all times:

$$(2.1)\qquad p' = 0 \quad \text{and} \quad w' = \underline{\Phi}_t + u'\underline{\Phi}_r + r^{-1}v'\underline{\Phi}_\theta \quad \text{on} \quad z = \underline{\Phi}$$

$$(2.2)\qquad w = 0 \quad \text{on} \quad z = 0$$

$$(2.3)\qquad u = 0 \quad \text{and} \quad u' = 0 \quad \text{on} \quad r = R$$

$$(2.4)\qquad p = p' \quad \text{and} \quad w - u\phi_r - r^{-1}v\phi_\theta = w' - u'\phi_r - r^{-1}v'\phi_\theta \quad \text{on} \quad z = \phi$$

Finally we shall impose, for reasons of mathematical simplicity, a kind of symmetry on the motions to be considered by requiring that

$$(2.5)\qquad u = 0 \quad \text{and} \quad u' = 0 \quad \text{on} \quad r = 0 \ .$$

The equations describing the incompressibility of the two liquids are

$$u_r + r^{-1}u + r^{-1}v_\theta + w_z = 0 \quad , \tag{2.6}$$

$$u'_r + r^{-1}u' + r^{-1}v'_\theta + w'_z = 0 \quad . \tag{2.7}$$

The conservation of momentum is similarly described by the equations

$$u_t + uu_r + r^{-1}vu_\theta + wu_z - r^{-1}v^2 + \alpha p_r = 0 \tag{2.8}$$

$$v_t + uv_r + r^{-1}vv_\theta + wv_z + r^{-1}uv + \alpha r^{-1}p_\theta = 0 \tag{2.9}$$

$$g + \alpha p_z = 0 \tag{2.10}$$

$$u'_t + u'u'_r + r^{-1}v'u'_\theta + w'u'_z - r^{-1}v'^2 + \alpha' p'_r = 0 \tag{2.11}$$

$$v'_t + u'v'_r + r^{-1}v'v'_\theta + w'v'_z + r^{-1}u'v' + \alpha' r^{-1}p'_\theta = 0 \tag{2.12}$$

$$g + \alpha' p'_z = 0 \quad . \tag{2.13}$$

(The hydrostatic approximation that vertical accelerations are small compared to the gravity acceleration has been used in equations (2.10) and (2.13).)

Although the differential equations and boundary conditions above completely describe the physical problem subject to appropriate initial conditions they are because of their number and complexity unsuitable for the further discussion of the problem. Our object will be to eliminate some of the dependent variables and incorporate the boundary conditions on the bottom, the interface and the free surface explicitly into a new set of equations. This object can be achieved by two different methods. The first, described in more detail in Appendix A, employs the additional

assumption, amply justified by the observations, that all the shear of the horizontal velocity components is concentrated in the interface. This method has the advantage that it leads to a set of six nonlinear differential equations with independent variables r, θ and t. These equations would be useful for numerical computation of the solution for given initial conditions and for analyzing stability of various types of motion.

The second method, to be described in Article 4, has the advantages for our present purposes that it does not require any additional assumptions and proceeds directly to a set of simplified linear equations.

§3. Equilibrium solutions of solid body rotation type.

It is readily verified that equations (2.1)-(2.13) allow exact solutions in which the radial and vertical components of velocity vanish identically and in which the remaining dependent variables are independent of the time and angle variables. For these solutions the pressure gradient in each liquid is in exact balance with the centripetal acceleration directed radially and the gravity acceleration directed vertically. The solutions are underdetermined to the extent that the transverse velocity distribution can be prescribed as an arbitrary function of the radius. Thus there exist, for example, double layer flows of the Rankine vortex type in which the velocity increases linearly from zero at the center to a maximum at some prescribed distance and then decreases like the inverse first power of the radius

out to the edge of the cylinder. All of the conceivably significant flows, however, share their most important properties with the simplest type of solution, that corresponding to solid body rotations of different speeds in the two layers. The rest of the discussion will be limited to such linear velocity profiles.

It is readily verified that corresponding to the prescribed transverse velocity distributions

$$v = \omega r \quad , \qquad v' = \omega' r \quad , \tag{3.1}$$

where ω and ω' are constant angular speeds, we have the height and pressure formulas

$$g\underline{\Phi}(r) = g\underline{\Phi}_o + \frac{1}{2}\omega'^2 r^2 \tag{3.2}$$

$$g\phi(r) = g\phi_o + \frac{1}{2}\frac{\rho\omega^2 - \rho'\omega'^2}{\rho - \rho'} r^2 \quad , \tag{3.3}$$

where $\underline{\Phi}_o = \underline{\Phi}(0)$ and $\phi_o = \phi(0)$ are prescribed constants, and

$$p'(r,z) = g\rho'(\underline{\Phi}-z) \tag{3.4}$$

$$p(r,z) = g\rho'(\underline{\Phi}-\phi) + g\rho(\phi-z) \quad . \tag{3.5}$$

Some interesting properties of the double layer solutions are easily deduced. Referring to equation (3.2) we note that the height of the free surface is independent of the densities in either layer, that it is a paraboloid dished upward and that, if account is taken of the experimental magnitude of the angular speed ω', the curvature is very small. From equation (3.3) we

note that the interface is likewise a paraboloid, but in contrast to the free surface, its slope and curvature depend strongly on the densities and on the angular speeds in the two layers. In the experiments the density difference is small and therefore if the angular speeds are not also nearly the same the interface will be strongly curved. Moreover, since the angular speed in the bottom layer can be either greater or less than that in the top layer, the paraboloid can be dished either up or down. These properties of the solution are in complete accord with the experimental observations.

For convenience later we introduce the thickness $\phi' = \underline{\Phi}-\phi$ of the upper layer given by the formula

$$g\phi'(r) = g\phi'_0 - \frac{1}{2}\frac{\rho(\omega^2 - \omega'^2)}{\rho - \rho'} r^2 \quad . \tag{3.6}$$

§4. Perturbations from solid rotation.

In order to find a simpler set of linear equations which describe motions close in some sense to the solid body rotations of the double layer we may take the first variation of the complete equations and boundary conditions of §2 and afterwards substitute for the unvaried quantities, wherever they occur, the solid body solutions of the last section. We denote a perturbation quantity by prefixing a "δ" to the symbol for the corresponding unvaried dependent variable. It is convenient also to introduce the notation for the linear operators $D = \partial/\partial t + \omega\,\partial/\partial\theta$ and $D' = \partial/\partial t + \omega'\,\partial/\partial\theta$. The result of the

operations described above on the equations (2.1)-(2.13) leads to the linear equations

(4.1) $\delta p' = 0$ and $\delta w' = D'\delta\underline{\Phi} + \underline{\Phi}_r \delta u'$ on $z = \underline{\Phi} + \delta\underline{\Phi}$

(4.2) $\delta w = 0$ on $z = 0$

(4.3) $\delta u = 0$ and $\delta u' = 0$ on $r = R$

(4.4) $p + \delta p = p' + \delta p'$ and

$$\delta w - \phi_r \delta u - \omega\delta\phi_\theta = \delta w' - \phi_r \delta u' - \omega'\delta\phi_\theta \quad \text{on} \quad z = \phi + \delta\phi$$

(4.5) $\delta u = 0$ and $\delta u' = 0$ on $r = 0$

(4.6) $\delta u_r + r^{-1}\delta u + r^{-1}\delta v_\theta + \delta w_z = 0$

(4.7) $\delta u'_r + r^{-1}\delta u' + r^{-1}\delta v'_\theta + \delta w'_z = 0$

(4.8) $D\delta u - 2\omega\delta v + a\delta p_r = 0$

(4.9) $D\delta v + 2\omega\delta u + ar^{-1}\delta p_\theta = 0$

(4.10) $a\delta p_z = 0$

(4.11) $D'\delta u' - 2\omega'\delta v' + a'\delta p'_r = 0$

(4.12) $D'\delta v' + 2\omega'\delta u' + a'r^{-1}\delta p'_\theta = 0$

(4.13) $a'\delta p'_z = 0$.

Some elementary operations lead at once to the results

(4.14) $a'\delta p' = g\delta\underline{\Phi} = g(\delta\phi + \delta\phi')$

(4.15) $$\alpha\delta p = g\beta(\delta\underline{\Phi} - \delta\phi) + g\delta\phi = g(\delta\phi + \beta\delta\phi')$$

where $\beta = \rho'/\rho = \alpha/\alpha'$. Also the velocity perturbations δu, δv, $\delta u'$ and $\delta v'$ are found to be independent of z so that the vertical velocity perturbations can be found explicitly in terms of the horizontal velocities:

(4.16) $$\delta w = -z(\delta u_r + r^{-1}\delta u + r^{-1}v_\theta)$$

(4.17) $$\delta w' = D'\delta\underline{\Phi} + (\underline{\Phi}-z)(\delta u'_r + r^{-1}\delta u' + r^{-1}v'_\theta) + \underline{\Phi}_r\delta u' \quad .$$

Finally the equations satisfied by the six perturbation quantities δu, δv, $\delta\phi$, $\delta u'$, $\delta v'$ and $\delta\phi'$ are

(4.18) $$D\delta u - 2\omega\delta v + g(\delta\phi + \beta\delta\phi')_r = 0$$

(4.19) $$D\delta v + 2\omega\delta u + gr^{-1}(\delta\phi + \beta\delta\phi')_\theta = 0$$

(4.20) $$D\delta\phi + \phi_r\delta u + \phi(\delta u_r + r^{-1}\delta u + r^{-1}\delta v_\theta) = 0$$

(4.21) $$D'\delta u' - 2\omega'\delta v' + g(\delta\phi + \delta\phi')_r = 0$$

(4.22) $$D'\delta v' + 2\omega'\delta u' + gr^{-1}(\delta\phi + \delta\phi')_\theta = 0$$

(4.23) $$D'\delta\phi' + \phi'_r\delta u' + \phi'(\delta u'_r + r^{-1}\delta u' + r^{-1}\delta v'_\theta) = 0$$

Summarizing the results just obtained, we may say that the assumptions of negligible vertical acceleration (hydrostatic balance) and small deviations from rotationally symmetric motion lead at once to the complete vertical homogeneity in each layer and the concentration of all velocity shear in the interface. We note also the characteristic property that in the lower layer

ascending motion is associated with convergence of the velocity field and descending motion with divergence; the magnitude of the vertical velocity is greatest at the interface. Finally it is worth pointing out the striking similarity between the equations (4.18)-(4.23) and the equations of dynamical meteorology.

§5. Progressing wave solutions.

The experiments of Fultz indicate plainly that steady wave motions of the two layers are possible or rather would be so if friction could be entirely eliminated. I therefore look for such progressing waves travelling with a uniform angular speed κ. The solutions, if they exist, will then depend periodically on the variable $\bar{\theta} = \theta - \kappa t$. If there are an integral number m of wave crests around the circumference we may write

$$\{\delta u, \delta u'\}(r,\theta,t) = \{U, U'\}(r) \sin m\bar{\theta} \tag{5.1}$$

$$\{\delta v, \delta v', \delta\phi, \delta\phi'\}(r,\theta,t) = \{V, V', H, H'\}(r) \cos m\bar{\theta} \ . \tag{5.2}$$

From now on we shall indicate derivatives with respect to r by a dot over the quantity. With the substitutions above the final equations of the last section become

$$m(\kappa - \omega)U + 2\omega V = g(\dot{H} + \beta\dot{H}') \tag{5.3}$$

$$2\omega U + m(\kappa - \omega)V = gmr^{-1}(H + \beta H') \tag{5.4}$$

$$m(\kappa - \omega)H + \dot{\phi}U + \phi(\dot{U} + r^{-1}U - mr^{-1}V) = 0 \tag{5.5}$$

$$(5.6)\qquad m(\kappa - \omega')U' + 2\omega'V' = g(\dot{H} + \dot{H}')$$

$$(5.7)\qquad 2\omega'U' + m(\kappa - \omega')V' = gmr^{-1}(H + H')$$

$$(5.8)\qquad m(\kappa - \omega')H' + \dot{\phi}'U' + \phi'(\dot{U}' + r^{-1}U' - mr^{-1}V') = 0 \quad .$$

If we make the abbreviations

$$(5.9)\qquad \gamma = m^2(\kappa - \omega)^2 - 4\omega^2$$

$$(5.10)\qquad \gamma' = m^2(\kappa - \omega')^2 - 4\omega'^2$$

then, providing $\sigma \neq 0$

$$(5.11)\qquad U = g\sigma^{-1}m[(\kappa - \omega)(\dot{H} + \beta\dot{H}') - 2\omega r^{-1}(H + \beta H')]$$

$$(5.12)\qquad V = -g\sigma^{-1}[2\omega(\dot{H} + \beta\dot{H}') - m^2(\kappa - \omega)r^{-1}(H + \beta H')]$$

$$(5.13)\qquad \dot{U} = g\sigma^{-1}m[(\kappa - \omega)(\ddot{H} + \beta\ddot{H}') - 2\omega r^{-1}(\dot{H} + \beta\dot{H}') + 2\omega r^{-2}(H + \beta H')]$$

These expressions may be substituted into equation (5.5) to obtain the differential equation

$$(5.14)\qquad \phi[(\ddot{H} + \beta\ddot{H}') + r^{-1}(\dot{H} + \beta\dot{H}') - m^2r^{-2}(H + \beta H')]$$

$$+ \dot{\phi}[(\dot{H} + \beta\dot{H}') - 2\omega(\kappa - \omega)^{-1}r^{-1}(H + \beta H')] + g^{-1}\sigma H = 0 \quad .$$

It is an observed fact in the Fultz experiments that when the density difference $\varepsilon = 1-\beta$ is small, the perturbations of the free surface are negligible compared to the perturbations of the interface. This property is not evident from the equations but it is physically plausible since the interface would be very sensitive to small accelerations when the lower liquid had

nearly zero weight. Referring to equations (5.6) and (5.7) we see that when H + H' is set equal to zero the velocities U' and V' satisfy a pair of homogeneous algebraic equations. For non-trivial solutions to exist the determinant of coefficients, i.e. σ' defined in equation (5.10), must vanish. We thus have the relations

$$m(k - \omega') = 2s\omega' \tag{5.15}$$

$$U' = -sV' \tag{5.16}$$

where $s = +1$ or -1.

The dependence of the velocity U' on r is determined after H has been found by an integration of equation (5.8) which reduces to the form

$$d(r^{ms+1}\,\phi' U') = 2s\omega' H r^{ms+1}\,dr \quad . \tag{5.17}$$

Finally equations (5.11)-(5.14) all simplify somewhat when use is made of the fact that $H+\beta H' = (1-\beta)H = \varepsilon H$.

§6. The hypergeometric equation.

In order to simplify equation (5.14) we make the substitution $H(r) = r^m Z(r)$, so that Z satisfies

$$\phi\ddot{Z} + r^{-1}[(2m+1)\phi + r\dot{\phi}]\dot{Z} + [\sigma/g\varepsilon + (m - \frac{2\omega}{k-\omega})r^{-1}\dot{\phi}]Z = 0 \quad . \tag{6.1}$$

Next write the equilibrium equation of the interface in the form

$$\phi = \phi_o(1 + kr^2) \quad , \qquad k = (\omega^2 - \beta\omega'^2)/2\varepsilon g\phi_o \tag{6.2}$$

and define the new independent variable

$$\xi = 1 - \phi/\phi_0 = -kr^2 \quad . \tag{6.3}$$

Then, with $Z(r) = \zeta(\xi)$, it is easily verified that ζ satisfies the hypergeometric equation

$$\xi(1-\xi)\ddot{\zeta} + [c - (a+b+1)\xi]\dot{\zeta} - ab\zeta = 0 \tag{6.4}$$

where

$$c = m+1 = a+b \quad , \qquad ab = \mu + \lambda$$
$$\mu = \sigma/2(\omega)^2 - \beta\omega'^2) \quad , \qquad \lambda = \frac{1}{2}(m - \frac{2\omega}{\kappa - \omega}) \tag{6.5}$$

§7. Solution of the boundary value problem.

Since $c = m+1$ is an integer, it is known from the general theory of the hypergeometric differential equation [8] that the appropriate pair of fundamental solutions of equation (6.5) is $\zeta_1(\xi) = F(a,b;c;\xi)$ and $\zeta_2(\xi) = F(a,b;1;1-\xi)$. The original interval for r is from $r = 0$ to $r = R$. The corresponding interval in ξ is therefore from $\xi = 0$ to $\xi = \xi_R = -kR^2 = 1 - \phi_R/\phi_0$. Of the two fundamental solutions only $\zeta_1(\xi)$ is regular at $\xi = 0$. The second solution behaves like ξ^{-m} near $\xi = 0$. Since the height and velocity perturbations are required to be bounded on the axis of the cylinder we discard the second solution.

Two cases must now be distinguished depending on whether the interface is curved upward ($k > 0$) or curved downward ($k < 0$). The former is called abnormal flow and the latter normal flow.

Since $\phi_o > 0$, sgn k = sgn $(\omega^2 - \beta\omega'^2)$. In abnormal flow $\phi_o << \phi_R$, whereas in normal flow the opposite is the case, $\phi_R << \phi_o$. The intervals of integration in the two cases are:

a) Abnormal flow: $-|\xi_R| \leq \xi \leq 0$, $|\xi_R| >> 1$

b) Normal flow: $0 \leq \xi \leq \xi_R < 1$.

The hypergeometric function $\zeta_1(\xi)$ is regular in both these intervals.

The boundary condition at the outer wall of the cylinder requires that the radial velocity must vanish. After the appropriate transformations of equation (5.11) the condition becomes

$$\xi\dot{\zeta}_1 = -\lambda\zeta_1 \qquad \text{at} \quad \xi = \xi_R \tag{7.1}$$

or by a well known property of the hypergeometric function, cf. Reference [8]

$$ab\xi_R F(a+1,b+1;\ m+2;\ \xi_R) = -(m+1)\lambda F(a,b;\ m+1;\ \xi_R) \quad . \tag{7.2}$$

Equation (7.2) is an eigenvalue equation to determine, for a given value of the integer m, the ratio of the speed in the lower liquid to the speed in the upper liquid. The value of the dimensionless Froude number defined by the relation

$$Fr = (\omega'^2 R^2/2\varepsilon g\phi_o)^{1/2} \tag{7.3}$$

and the number $s = \pm 1$ appear as parameters in equation (7.2) whose values must be chosen beforehand.

The calculation of the eigenvalues and subsequently the heights and velocities is a difficult problem in itself. The procedures which were used and a detailed analysis of the results will be given in a separate report. Here I shall only discuss the various cases and summarize the eigenvalues.

Consider first the case of abnormal flow in which the denser liquid is piled up near the cylinder rim. We observed earlier that for this case $\omega/\omega' > \sqrt{\beta} > 1$. From equation (5.15) the wave propagation speed is $k/\omega' = 1 + \frac{2s}{m}$. Thus if $s = +1$, $k/\omega' > 1$, i.e. the waves are progressing; the relative wave speed is highest for small integer values of m and decreases monotonely with increasing number of waves. If $s = -1$, $k/\omega' < 1$, i.e. the waves are retrograde; for low integer values of m the speed is likewise low and increases with m. In abnormal flow both progressing and retrograde waves are mathematically possible. The existence of retrograde waves in the laboratory has not yet been established so far as I know. Tables I and II below give the relative wave speeds and the lower liquid relative speed for various wave numbers and Froude numbers appropriate to abnormal flow both progressing and retrograde.

m	ω/ω'			k/ω'
	Fr = 10	Fr = 14.14	Fr = 17.32	
3	1.775	1.794	1.802	1.667
4	1.640	1.630	1.637	1.500
5	1.545	1.550	1.550	1.400
6	1.420	1.435	1.438	1.333
7	1.350	1.361	1.363	1.283
8	1.300	1.308	1.310	1.250

Table I. Progressing waves, abnormal flow.

m	ω/ω'			k/ω'
	Fr = 10	Fr = 14.14	Fr = 17.32	
3	2.993	2.671	2.518	0.333
4	2.246	2.089	2.014	0.500
5	2.198	2.093	2.043	0.600
6	2.243	2.163	2.127	0.667
7	2.302	2.238	2.210	0.717
8	2.359	2.306	2.282	0.750

Table II. Retrograde waves, abnormal flow.

In the case of normal flow the denser liquid is piled up near the axis of the cylinder and the condition is that $\omega/\omega' < \sqrt{\beta} \simeq 1$. As before the wave speed condition is that $k/\omega' = 1 + \frac{2s}{m}$. Heuristically it would seem rather improbable that wave speeds greater than the liquid speed in both layers could exist and indeed no roots of the transcendental equation (7.2) could be found corresponding to s = +1. The normal progressing waves correspond to s = -1. Table III presents the values of relative wave speed and relative lower liquid speed.

m	ω/ω'			k/ω'
	Fr = 1.1	Fr = 1.2	Fr = 1.3	
3	xxx	xxx	xxx	0.333
4	.433	xxx	xxx	0.500
5	.516	.562	xxx	0.600
6	.589	.589	.644	0.667
7	.644	.641	.648	0.717

Table III. Progressing waves, normal flow.

§8. Appendix A.

If in the equations of §2 we make the a priori assumption that the vertical shear is concentrated entirely in the interface, i.e. the horizontal velocity components are constant separately in each layer but have a jump discontinuity at the front, then the conservation of momentum equations simplify. They may be written

$$\bar{D}u - r^{-1}v^2 + \alpha p_r = 0 \tag{8.1}$$

$$\bar{D}v + r^{-1}uv + \alpha r^{-1} p_\theta = 0 \tag{8.2}$$

$$\bar{D}'u' - r^{-1}v'^2 + \alpha' p'_r = 0 \tag{8.3}$$

$$\bar{D}'v' + r^{-1}u'v' + \alpha' r^{-1} p'_\theta = 0 \tag{8.4}$$

where we have introduced the nonlinear derivative operators $\bar{D} \equiv (\partial/\partial t + u\,\partial/\partial r + r^{-1}v\,\partial/\partial\theta)$ and $\bar{D}' \equiv (\partial/\partial t + u'\,\partial/\partial r + r^{-1}v'\,\partial/\partial\theta)$.

The equations of mass conservation in the two layers can be written immediately in the integrated form

$$w = -z\triangle \tag{8.5}$$

$$w' = -z\triangle' + \phi(\triangle' - \triangle) + [u]\phi_r + r^{-1}[v]\phi_\theta \tag{8.6}$$

where $\triangle$ and $\triangle'$ stand for the horizontal velocity divergences in the two layers and $[f] = f' - f$.

At the interface and the free surface the vertical velocities can be written, by using the kinematic boundary conditions, as

(8.7) $$w\ (z = \phi) = \bar{D}\phi$$

(8.8) $$w'\ (z = \underline{\Phi}) = \bar{D}'\underline{\Psi} = \bar{D}'\phi + \bar{D}'\phi' \quad .$$

Then it follows from these equations and equations (8.5) and (8.6) applied at $z = \phi$ and $z = \underline{\Phi}$ respectively that

(8.9) $$\bar{D}\phi + \phi(u_r + r^{-1}u + r^{-1}v_\theta) = 0$$

(8.10) $$\bar{D}'\phi' + \phi'(u'_r + r^{-1}u' + r^{-1}v'_\theta) = 0 \quad .$$

Integration of the hydrostatic pressure equations gives as before

(8.11) $$\alpha' p' = g(\phi + \phi' - z)$$

(8.12) $$\alpha p = g(\phi + \beta\phi' - z)$$

which may be substituted in equations (8.1)-(8.4) to complete the system.

It is sometimes useful to consider the vorticities in the two layers, defined by

(8.13) $$\Omega = v_r + r^{-1}v - r^{-1}u_\theta$$

(8.14) $$\Omega' = v'_r + r^{-1}v' - r^{-1}u'_\theta \quad .$$

Then it is easily deduced from the preceding results that the potential vorticities Ω/ϕ and Ω'/ϕ' are conserved in the motion, i.e.

(8.15) $$\bar{D}(\Omega/\phi) = 0$$

(8.16) $$\bar{D}'(\Omega'/\phi') = 0 \quad .$$

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